6-7 The Remainder and Factor Theorems

Main Ideas

- Evaluate functions using synthetic substitution.
- Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

New Vocabulary

synthetic substitution
depressed polynomial

GET READY for the Lesson

The number of international travelers to the United States since 1986 can be modeled by the equation $T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9$, where x is the number of years since 1986 and $T(x)$ is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function by substituting 20 for x, or you can use synthetic substitution.

Synthetic Substitution Synthetic division can be used to find the value of a function. Consider the polynomial function $f(a) = 4a^2 - 3a + 6$. Divide the polynomial by $a - 2$.

Method 1 Long Division

$$\begin{array}{r} 4a + 5 \\ a - 2 \overline{)\,4a^2 - 3a + 6} \\ \underline{4a^2 - 8a} \\ 5a + 6 \\ \underline{5a - 10} \\ 16 \end{array}$$

Method 2 Synthetic Division

$$\begin{array}{r|rrr} 2 & 4 & -3 & 6 \\ & & 8 & 10 \\ \hline & 4 & 5 & 16 \end{array}$$

Compare the remainder of 16 to $f(2)$.

$$\begin{aligned} f(2) &= 4(2)^2 - 3(2) + 6 && \text{Replace } a \text{ with 2.} \\ &= 16 - 6 + 6 && \text{Multiply.} \\ &= 16 && \text{Simplify.} \end{aligned}$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This illustrates the **Remainder Theorem**.

KEY CONCEPT *Remainder Theorem*

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

Dividend	equals	quotient	times	divisor	plus	remainder.
$f(x)$	$=$	$q(x)$	$\cdot$	$(x - a)$	$+$	$f(a)$,

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

EXAMPLE 1 Synthetic Substitution

If $f(x) = 2x^4 - 5x^2 + 8x - 7$, find $f(6)$.

Method 1 Synthetic Substitution

By the Remainder Theorem, $f(6)$ should be the remainder when you divide the polynomial by $x - 6$.

$$\begin{array}{r|rrrrr} 6 & 2 & 0 & -5 & 8 & -7 \\ & & 12 & 72 & 402 & 2460 \\ \hline & 2 & 12 & 67 & 410 & 2453 \end{array}$$

Notice that there is no x^3 term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, $f(6) = 2453$.

Method 2 Direct Substitution

Replace x with 6.

$f(x) = 2x^4 - 5x^2 + 8x - 7$ Original function

$f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7$ Replace x with 6.

$= 2592 - 180 + 48 - 7$ or 2453 Simplify.

By using direct substitution, $f(6) = 2453$. Both methods give the same result.

CHECK Your Progress

1A. If $f(x) = 3x^3 - 6x^2 + x - 11$, find $f(3)$.

1B. If $g(x) = 4x^5 + 2x^3 + x^2 - 1$, find $f(-1)$.

Factors of Polynomials The synthetic division below shows that the quotient of $x^4 + x^3 - 17x^2 - 20x + 32$ and $x - 4$ is $x^3 + 5x^2 + 3x - 8$.

$$\begin{array}{r|rrrrr} 4 & 1 & 1 & -17 & -20 & 32 \\ & & 4 & 20 & 12 & -32 \\ \hline & 1 & 5 & 3 & -8 & 0 \end{array}$$

Study Tip

Depressed Polynomial

A *depressed polynomial* has a degree that is one less than the original polynomial.

When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial**. From the results of the division and by using the Remainder Theorem, we can make the following statement.

Dividend	equals	quotient	times	divisor	plus	remainder.
$x^4 + x^3 - 17x^2 - 20x + 32$	$=$	$(x^3 + 5x^2 + 3x - 8)$	$\cdot$	$(x - 4)$	$+$	0

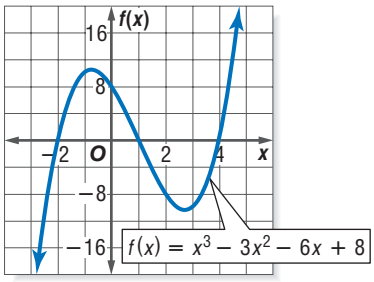
Since the remainder is 0, $f(4) = 0$. This means that $x - 4$ is a factor of $x^4 + x^3 - 17x^2 - 20x + 32$. This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

KEY CONCEPT *Factor Theorem*

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$, or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

Suppose you wanted to find the factors of $x^3 - 3x^2 - 6x + 8$. One approach is to graph the related function, $f(x) = x^3 - 3x^2 - 6x + 8$. From the graph, you can see that the graph of $f(x)$ crosses the x-axis at -2, 1, and 4. These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.



$$f(x) = [x - (-2)](x - 1)(x - 4)$$
$$= (x + 2)(x - 1)(x - 4)$$

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed, and once graphed, the exact zeros are often difficult to determine.

EXAMPLE 2 Use the Factor Theorem

Show that $x + 3$ is a factor of $x^3 + 6x^2 - x - 30$. Then find the remaining factors of the polynomial.

The binomial $x + 3$ is a factor of the polynomial if -3 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

-3	1	6	-1	-30
		-3	-9	30
	1	3	-10	0

Since the remainder is 0, $x + 3$ is a factor of the polynomial. The polynomial $x^3 + 6x^2 - x - 30$ can be factored as $(x + 3)(x^2 + 3x - 10)$. The polynomial $x^2 + 3x - 10$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 + 3x - 10 = (x - 2)(x + 5)$ Factor the trinomial.

So, $x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5)$.

Study Tip

Factoring

The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

CHECK Your Progress

2. Show that $x - 2$ is a factor of $x^3 - 7x^2 + 4x + 12$. Then find the remaining factors of the polynomial.

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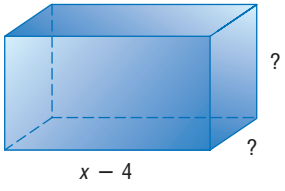
EXAMPLE 3 Find All Factors

GEOMETRY The volume of the rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures.



The volume of a rectangular prism is $\ell \times w \times h$.

You know that one measure is $x - 4$, so $x - 4$ is a factor of $V(x)$.

4	1	3	-36	32
		4	28	-32
	1	7	-8	0

The quotient is $x^2 + 7x - 8$. Use this to factor $V(x)$.

Extra Examples at algebra2.com

$V(x) = x^3 + 3x^2 - 36x + 32$ Volume function

$= (x - 4)(x^2 + 7x - 8)$ Factor.

$= (x - 4)(x + 8)(x - 1)$ Factor the trinomial $x^2 + 7x - 8$.

So, the missing measures of the prism are $x + 8$ and $x - 1$.

CHECK Your Progress

3. The volume of a rectangular prism is given by $V(x) = x^3 + 7x^2 - 36$. Find the expressions for the dimensions of the prism.

CHECK Your Understanding

Example 1 (p. 357)

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

1. $f(x) = x^3 - 2x^2 - x + 1$

2. $f(x) = 5x^4 - 6x^2 + 2$

For Exercises 3–5, use the following information.

The projected sales of e-books in millions of dollars can be modeled by the function $S(x) = -17x^3 + 200x^2 - 113x + 44$, where x is the number of years since 2000.

3. Use synthetic substitution to estimate the sales for 2008.

4. Use direct substitution to evaluate $S(8)$.

5. Which method—synthetic substitution or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer.

Examples 2, 3 (pp. 358–359)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. $x^3 - x^2 - 5x - 3; x + 1$

7. $x^3 - 3x + 2; x - 1$

8. $6x^3 - 25x^2 + 2x + 8; 3x - 2$

9. $x^4 + 2x^3 - 8x - 16; x + 2$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–17	1
18–29	2, 3
30–33	3

Use synthetic substitution to find $g(3)$ and $g(-4)$ for each function.

10. $g(x) = x^2 - 8x + 6$

11. $g(x) = x^3 + 2x^2 - 3x + 1$

12. $g(x) = x^3 - 5x + 2$

13. $g(x) = x^4 - 6x - 8$

14. $g(x) = 2x^3 - 8x^2 - 2x + 5$

15. $g(x) = 3x^4 + x^3 - 2x^2 + x + 12$

16. $g(x) = x^5 + 8x^3 + 2x - 15$

17. $g(x) = x^6 - 4x^4 + 3x^2 - 10$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

18. $x^3 + 2x^2 - x - 2; x - 1$

19. $x^3 - x^2 - 10x - 8; x + 1$

20. $x^3 + x^2 - 16x - 16; x + 4$

21. $x^3 - 6x^2 + 11x - 6; x - 2$

22. $2x^3 - 5x^2 - 28x + 15; x - 5$

23. $3x^3 + 10x^2 - x - 12; x + 3$

24. $2x^3 + 7x^2 - 53x - 28; 2x + 1$

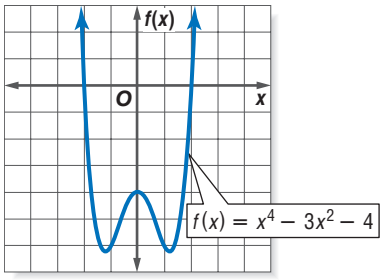
25. $2x^3 + 17x^2 + 23x - 42; 2x + 7$

26. $x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1$

27. $16x^5 - 32x^4 - 81x + 162; x - 2$

28. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors.

29. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial.



Cross-Curricular Project

Changes in world population can be modeled by a polynomial equation. Visit algebra2.com to continue work on your project.

BOATING **For Exercises 30 and 31, use the following information.**

A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in feet per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

30. Find the speed of the boat at 1, 2, and 3 seconds.

31. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find $f(6)$ and explain what this means.

ENGINEERING **For Exercises 32 and 33, use the following information.**

When a certain type of plastic is cut into sections, the length of each section determines its strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

32. Show that $x - 5$ is a factor of the polynomial function.

33. Are there other lengths of plastic that are extremely weak? Explain your reasoning.

Find values of k so that each remainder is 3.

34. $(x^2 - x + k) \div (x - 1)$

35. $(x^2 + kx - 17) \div (x - 2)$

36. $(x^2 + 5x + 7) \div (x + k)$

37. $(x^3 + 4x^2 + x + k) \div (x + 2)$

PERSONAL FINANCE **For Exercises 38–41, use the following information.**

Zach has purchased some home theater equipment for $2000, which he is financing through the store. He plans to pay $340 per month and wants to have the balance paid off after six months. The formula $B(x) = 2000x^6 - 340(x^5 + x^4 + x^3 + x^2 + x + 1)$ represents his balance after six months if x represents 1 plus the monthly interest rate (expressed as a decimal).

38. Find his balance after 6 months if the annual interest rate is 12%. (*Hint*: The monthly interest rate is the annual rate divided by 12, so $x = 1.01$.)

39. Find his balance after 6 months if the annual interest rate is 9.6%.

40. How would the formula change if Zach wanted to pay the balance in five months?

41. Suppose he finances his purchase at 10.8% and plans to pay $410 every month. Will his balance be paid in full after five months?

EXTRA PRACTICE
See pages 904, 931.
Math nline
Self-Check Quiz at algebra2.com

H.O.T. Problems

42. **OPEN ENDED** Give an example of a polynomial function that has a remainder of 5 when divided by $x - 4$.

43. **REASONING** Determine the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.

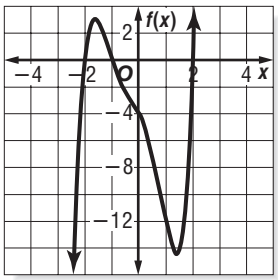
−2	1	0	6	32
		−2	4	−20
	1	−2	10	12

44. CHALLENGE Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

45. *Writing in Math* Use the information on page 356 to explain how you can use the Remainder Theorem to evaluate polynomials. Include an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution. Evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

STANDARDIZED TEST PRACTICE

46. ACT/SAT Use the graph of the polynomial function at the right. Which is *not* a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?



A $(x - 2)$

B $(x + 2)$

C $(x - 1)$

D $(x + 1)$

47. REVIEW The total area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

F $(5a^2 + 4b)(5a^2 + 4b)$

G $(5a^2 + 4b)(5a^2 - 4b)$

H $(5a - 4b)(5a - 4b)$

J $(5a + 4b)(5a - 4b)$

Spiral Review

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

48. $7xy^3 - 14x^2y^5 + 28x^3y^2$

49. $ab - 5a + 3b - 15$

50. $2x^2 + 15x + 25$

51. $c^3 - 216$

Graph each function by making a table of values. (Lesson 6-5)

52. $f(x) = x^3 - 4x^2 + x + 5$

53. $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

54. CITY PLANNING City planners have laid out streets on a coordinate grid before beginning construction. One street lies on the line with equation $y = 2x + 1$. Another street that intersects the first street passes through the point $(2, -3)$ and is perpendicular to the first street. What is the equation of the line on which the second street lies? (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL Find the exact solutions of each equation by using the Quadratic Formula. (Lesson 5-6)

55. $x^2 + 7x + 8 = 0$

56. $3x^2 - 9x + 2 = 0$

57. $2x^2 + 3x + 2 = 0$